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A MATHEMATICAL EXAMINATION OF SPIRALED COMPOUND CURVES

by T. F. Hickerson, M. ASCE

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A MATHEMATICAL EXAMINATION OF SPIRALED COMPOUND CURVES

H. F. Hickerson, M. ASCE

SYNOPSIS

In considering spiraled compound curves (see Figs. 1 and 2), one is confronted with the question: What are the limits of spiral length and total change in curvature beyond which desirable accuracy is not attainable. The reason for this doubt arises from the usual assumption based partly on intuition - that the total radial separation of the two branches of the compound curve depends on the nominal "spiral angle", a quantity analogous to the central angle of a spiral joining a straight line with a circular curve.

It will be shown that the conventional procedure agrees closely with rigorous theory for the ordinary range of values. This is to be expected since a transition spiral is a curve of uniformly changing curvature, hence it should diverge in angle and offset, for a given distance, at about the same rate from the osculating circle as from the initial tangent.

INTRODUCTION

Referring to Figs. 1 and 2, let D_1 and R_1 equal, respectively, the degree of curve and radius of the flatter curve subtending central angle Δ_1; while D_2 and R_2 apply to the sharper curve subtending angle Δ_2; and let the total central angle = Δ (= $\Delta_1 + \Delta_2$). Also

L = AP = length of any spiral arc;
L_S = AB_1 = total length of spiral;
β = central angle of arc AP ($\beta = \Delta$, for arc AB_1);
D = degree of curve of the spiral at any point (as P);
θ_S = nominal spiral angle;
c = chord AB_1 = spiral long chord;
p = total radial shift at C (the P.C.C.), p = CC_1 = OO_2 = BB_1;
x,y = coordinates of any point P of the spiral (see (b) of Figs. 1 and 2);
x_1 = AG, y_1 = GB_1
T_1 and T_2 = tangent distances AV and VB_1, respectively, where T_1 is the long tangent and T_2 is the short tangent.

CASE I: <u>Sharper branch shifted radially inward at C, Fig. 1.</u>

Since it is the function of the transition spiral to change the degree of curve uniformly as the arc distance from A (the C.S.) to any point P, we have

$$D = D_1 + {}^{L}/_{L_s} (D_2 - D_1). \tag{a}$$

From the differential sector at point P (see (b) of Fig. 1), we have

$$d\beta = \frac{dL}{R} = \frac{D}{100}\, dL, \tag{b}$$

where R (feet) = $\frac{100}{D}$, D being measured in radians. Substituting (a) in (b), we get

$$d\beta = \frac{1}{100}\left[D_1 + {}^{L}/_{L_s}(D_2 - D_1)\right] dL. \qquad \text{(c)}$$

Integrating, we get

$$\beta = \frac{1}{100}\left[D_1 L + \left(\frac{D_2 - D_1}{L_s}\right)\frac{L^2}{2}\right] = \frac{D_1 L}{100} + \left({}^{L}/_{L_s}\right)^2 \times \theta_s, \qquad \text{(d)}$$

where $\theta_s = \frac{L_s}{200}(D_2 - D_1)$ = nominal spiral angle.

Noting that the coordinate axes coincide with AG and O_1A respectively, we have (see Fig. 1b),

$$dx = dL \cos\beta = dL\left[1 - \frac{\beta^2}{2!} + \frac{\beta^4}{4!} - \ldots\ldots\right], \qquad \text{(e)}$$

and

$$dy = dL \sin\beta = dL\left[\beta - \frac{\beta^3}{3!} + \frac{\beta^5}{5!} - \ldots\ldots\right], \qquad \text{(f)}$$

where $\cos\beta$ and $\sin\beta$ are expanded into series.

Substituting (d) in (e), integrating, and reducing, we get

$$x_1 = \int_0^{L_s}\left[1 - \frac{\beta^2}{2!} + \frac{\beta^4}{4!} - \ldots\ldots\right] dL =$$

$$\int_0^{L_s} dL - \frac{1}{2!}\int_0^{L_s}\left[\frac{D_1 L}{100} + \left({}^{L}/_{L_s}\right)^2 \theta_s\right]^2 \times dL + \frac{1}{4!}\int_0^{L_s}\left[\frac{D_1 L}{100} + \left({}^{L}/_{L_s}\right)^2 \theta_s\right]^4 dL = L_s - \frac{1}{2}L_s\,\theta_s^2\left[\frac{1}{3}\left(\frac{D_1}{\theta_s}\right)^2\left(\frac{L_s}{100}\right)^2 +\right.$$

$$\left.\frac{1}{2}\,\frac{D_1}{\theta_s}\,\frac{L_s}{100} + \frac{1}{5}\right] + \frac{1}{24}L_s\,\theta_s^4\left[\frac{1}{5}\left(\frac{D_1}{\theta_s}\right)^4\left(\frac{L_s}{100}\right)^4 +\right.$$

$$\left.\frac{2}{3}\left(\frac{D_1}{\theta_s}\right)^3\left(\frac{L_s}{100}\right)^3 + \frac{6}{7}\left(\frac{D_1}{\theta_s}\right)^2\left(\frac{L_s}{100}\right)^2 + \frac{1}{2}\,\frac{D_1}{\theta_s}\,\frac{L_s}{100} + \frac{1}{9}\right],$$

..........................(g)

in which θ_s is to be expressed in radians.

Factoring, and replacing radians with degrees, (g) becomes

$$x_1 = AG = L_s\left[1 - \theta_s^2\left(\frac{0.1523087}{10^3}\right)\left[\frac{1}{3}\left(\frac{D_1}{\theta_s}\right)^2\left(\frac{L_s}{100}\right)^2 + \frac{1}{2}\,\frac{D_1}{\theta_s}\,\frac{L_s}{100} + \frac{1}{5}\right] + \theta_s^4\left(\frac{0.3866324}{10^8}\right)\left[\frac{1}{5}\left(\frac{D_1}{\theta_s}\right)^4\left(\frac{L_s}{100}\right)^4 + \frac{2}{3}\left(\frac{D_1}{\theta_s}\right)^3\left(\frac{L_s}{100}\right)^3 + \frac{6}{7}\left(\frac{D_1}{\theta_s}\right)^2\left(\frac{L_s}{100}\right)^2 + \frac{1}{2}\,\frac{D_1}{\theta_s}\,\frac{L_s}{100} + \frac{1}{9}\right]\right]. \quad (1)$$

Substituting (d) in (f) and proceeding in a similar manner, we get

$$y_1 = GB_1 = L_s\left[\theta_s\,(0.01745329)\left[\frac{1}{2}\,\frac{D_1}{\theta_s}\,\frac{L_s}{100} + \frac{1}{3}\right] - \theta_s^3\left(\frac{0.8860964}{10^6}\right)\left[\frac{1}{4}\left(\frac{D_1}{\theta_s}\right)^3\left(\frac{L_s}{100}\right)^3 + \left(\frac{3}{5}\right)\left(\frac{D_1}{\theta_s}\right)^2\left(\frac{L_s}{100}\right)^2 + \frac{1}{2}\,\frac{D_1}{\theta_s}\,\frac{L_s}{100} + \frac{1}{7}\right] + \theta_s^5\left(\frac{0.13496017}{10^{10}}\right)\left[\frac{1}{6}\left(\frac{D_1}{\theta_s}\right)^5\left(\frac{L_s}{100}\right)^5 + \frac{5}{7}\left(\frac{D_1}{\theta_s}\right)^4\left(\frac{L_s}{100}\right)^4 + \frac{10}{8}\left(\frac{D_1}{\theta_s}\right)^3\left(\frac{L_s}{100}\right)^3 + \frac{10}{9}\left(\frac{D_1}{\theta_s}\right)^2\left(\frac{L_s}{100}\right)^2 + \frac{5}{10}\,\frac{D_1}{\theta_s}\,\frac{L_s}{100} + \frac{1}{11}\right]\right], \ldots\ldots (2)$$

in which the angles are expressed in degrees.

The parts of the spiral may now be found. Thus:

ϕ_1 = total deflection angle from A to B_1 ,

$$\phi_1 = \text{arc tan}\ \frac{y_1}{x_1}\ ; \qquad (3)$$

$$\phi_2 = \Delta - \phi_1\ ; \qquad (4)$$

$$c = \text{spiral "long chord"} = \frac{y_1}{\sin \phi_1}\ . \qquad (5)$$

Next we have:

$$T_1 = AV = \text{long tangent} = AG - VG = x_1 - y_1 \cot \Delta \; ; \qquad (6)$$

$$T_2 = VB_1 = \text{short tangent} = \frac{GB_1}{\sin \Delta} = \frac{y_1}{\sin \Delta} \; ; \qquad (7)$$

In Fig. 1a, let verticals through B_1 and B intersect a horizontal line through C at H and M respectively. Also, let a horizontal line through B intersect the vertical GB_1 at K.

Then

$$y_1 = GB_1 = GH + HK + KB_1 \, , \qquad (h)$$

where $GH = FC = R_1 \text{ vers } \Delta_1$,

$HK = MB = BC \sin BCM = 2R_2\left[\sin\left(\frac{1}{2}\Delta_2\right)\sin\left(\Delta_1 + \frac{1}{2}\Delta_2\right)\right]$,

and $KB_1 = BB_1 \cos \Delta_1 = p \cos \Delta_1$.

Substituting in (h), and reducing, we get

$$p = \frac{y_1 - R_1 \text{ vers } \Delta_1 - 2R_2 \sin \frac{1}{2}\Delta_2 \sin(\Delta_1 + \frac{1}{2}\Delta_2).}{\cos \Delta_1} \qquad (8)$$

Eqs. (1-8) are basic formulas giving the various parts of a spiraled compound curve.

<u>CONVENTIONAL</u> <u>METHOD</u>: An independent, but slightly approximate, set of formulas will now be derived on the basis of the radial shift p found in current handbooks on route surveying when the "equivalent" spiral angle and length of spiral are given. This tabular value of p is compiled from the formula:

$$p = L_s\left[0.0014544410\,\theta_s - \frac{0.1582315\,\theta_s^3}{10^7} - \frac{0.1022426\,\theta_s^5}{10^{12}} - \frac{0.40785\,\theta_s^7}{10^{18}} - \ldots\ldots\ldots\right], \quad (\theta_s \text{ in degrees}) \qquad (8)'$$

With O_1A as zero azimuth (north), treat $AO_1O_2B_1A$ (Fig. 1) as a closed traverse. Accordingly, y = d cos A and x = d sin A, where d and A represent distance and azimuth respectively, as indicated below.

Point	Azimuth	Distance	y	x
A(C.S.)	180°	R_1	$-R_1$	O
O_1	Δ_1	O_1O_2	$(R_1-R_2-p)\cos\Delta_1$	$(R_1-R_2-p)\sin\Delta_1$
O_2	Δ	R_2	$R_2\cos\Delta$	$R_2\sin\Delta$
B_1	$270°+\phi_1$	c	$c\sin\phi_1$	$-c\cos\phi_1$

From $\Sigma y = 0$ and $\Sigma x = 0$, we get

$$c \sin \phi_1 = R_1 - (R_1 - R_2 - p) \cos \Delta_1 - R_2 \cos \Delta \qquad \text{(i)}$$

$$c \cos \phi_1 = (R_1 - R_2 - p) \sin \Delta_1 + R_2 \sin \Delta \qquad \text{(j)}$$

Let

$$N_1 = R_1 - (R_1 - R_2 - p) \cos \Delta_1 - R_2 \cos \Delta \qquad \text{(k)}$$

$$B_1 = (R_1 - R_2 - p) \sin \Delta_1 + R_2 \sin \Delta. \qquad \text{(l)}$$

Then on dividing (i) by (j), we get

$$\tan \phi_1 = \frac{N_1}{B_1}, \text{ from which}$$

$$\phi_1 = \text{arc} \tan \frac{N_1}{B_1} \qquad (9)$$

This value of ϕ_1 should agree closely with the spiral deflection formula:

$$\frac{L_S D_1}{200} + \frac{\theta_S}{3}.$$

From (j), we get

$$c = \frac{B_1}{\cos \phi_1}. \qquad (10)$$

Applying the law of sines (see Fig. 1a), we have

$$T_2 = \frac{c}{\sin \Delta} \times \sin \phi_1 = \frac{N_1}{\sin \Delta}. \qquad (11)$$

Since $T_1 + VG = T_1 + T_2 \cos\Delta = c \cos \phi_1$, we get

$$T_1 = B_1 - T_2 \cos \Delta = B_1 - N_1 \cot \Delta. \qquad (12)$$

As an independent check,

$$c = T_1 \cos \phi_1 + T_2 \cos \phi_2. \qquad \text{(m)}$$

EXAMPLE 1. Given $D_1 = 6°$, $D_2 = 10°$, $L_S = 300$ ft. Required: The parts of a transition spiral (Fig. 1) by both the exact and the conventional methods.

SOLUTION: The nominal spiral angle = $\theta_S = \frac{300}{200}(10-6) = 6°$; $\frac{\theta_S}{D_1} = 1$.

EXACT METHOD: Applying Eqs. 1 and 2, we get $x_1 = 292.33$ ft., $y_1 = 56.808$ ft.; from which (by Eqs. 2-8): $\phi_1 = 10°59'49''$, $\phi_2 = 13°00'11''$, $c = 297.80$ ft., $T_1 = 164.74$ ft., $T_2 = 139.67$ ft., and $p = 2.603$ ft.

CONVENTIONAL METHOD: With $\theta_S = 6^0$ and $L_S = 300$ ft., $p = 300 \times 0.00872 = 2.616$ ft., in which 0.00872 is a unit value of p (found in tables) as if the spiral joined a straight line with a circular curve. Accordingly, by Eqs. 9-12, $\phi_1 = 10^059'51''$, $\phi_2 = 13^000'09''$, c = 297.86 ft., $T_2 = 139.70$ ft., and $T_1 = 164.76$ ft.

EXAMPLE 2: Given $D_1 = 10^0$, $D_2 = 20^0$, $L_S = 200$ ft. Required: The parts of a transition spiral (Fig. 1) by both the exact and the conventional method.

SOLUTION: The nominal spiral angle $= \frac{200}{200}(20-10) = 10^0$; $\theta_S/D_1 = 1$.

EXACT METHOD: Applying Eqs. 1 and 2, we get $x_1 = 192.38$ ft., $y_1 = 45.569$ ft.; from which (Eqs. 2-8), $\phi_1 = 13^019'32''$, $\phi_2 = 16^040'28''$, c = 197.71 ft., $T_1 = 113.46$ ft., $T_2 = 91.14$ ft., p = 2,879 ft.

CONVENTIONAL METHOD: With $\theta_S = 10^0$ and $L_S = 200$ ft., $p = 200 \times 0.01453 = 2.906$ ft., in which 0.01453 is a unit value of p (found in tables) as if the spiral joined a straight line with a circular curve. Accordingly, by Eqs. 9-12, $\phi_1 = 13^019'36''$, $\phi_2 = 16^040'24''$, c = 197.81 ft., $T_2 = 91.19$ ft., and $T_1 = 113.51$ ft.

CASE II: Flatter branch shifted radially outward at C, Fig. 2.

In this case, the degree of curve decreases uniformly as the arc distance from A (the C.S.), hence

$$D = D_2 - {}^L/_{L_s}\,(D_2 - D_1). \qquad (n)$$

Substituting (n) in (b), integrating, and reducing, we get

$$\beta = \frac{D_2 L}{100} - \left({}^L/_{L_s}\right)^2 \theta_s, \qquad (o)$$

where $\theta_s = \frac{L_s}{200}(D_2 - D_1)$, as in Case I.

Substituting, in turn, (o) in (e) and (f), integrating, and reducing, we get

$$x_1 = AG = L_s\left[1 - \theta_s^2\left(\frac{0.1523087}{10^3}\right)\left[\frac{1}{3}\left(\frac{D_2}{\theta_s}\right)^2\left(\frac{L_s}{100}\right)^2 - \frac{1}{2}\,\frac{D_2}{\theta_s}\times\frac{L_s}{100}+\frac{1}{5}\right] + \theta_s^4\left(\frac{0.3866324}{10^8}\right)\left[\frac{1}{5}\left(\frac{D_2}{\theta_s}\right)^4\left(\frac{L_s}{100}\right)^4 - \frac{2}{3}\left(\frac{D_2}{\theta_s}\right)^3\left(\frac{L_s}{100}\right)^3 + \frac{6}{7}\left(\frac{D_2}{\theta_s}\right)^2\left(\frac{L_s}{100}\right)^2 - \frac{1}{2}\,\frac{D_2}{\theta_s}\times\frac{L_s}{100}+\frac{1}{9}\right]\right]; \qquad (13)$$

and

$$y_1 = GB_1 = L_s\left[\theta_s\ (0.01745329)\left[\frac{1}{2}\,\frac{D_2}{\theta_s}\,\frac{L_s}{100} - \frac{1}{3}\right] - \theta_s^3\left(\frac{0.8860964}{10^6}\right)\left[\frac{1}{4}\left(\frac{D_2}{\theta_s}\right)^3\left(\frac{L_s}{100}\right)^3 - \frac{3}{5}\left(\frac{D_2}{\theta_s}\right)^2\left(\frac{L_s}{100}\right)^2 + \right.\right.$$

$$\frac{1}{2}\ \frac{D_2}{\theta_s}\ \frac{L_s}{100} - \frac{1}{7}\Bigg] + \theta_s^5\left(\frac{0.13496017}{10^{10}}\right)\Bigg[\frac{1}{6}\left(\frac{D_2}{\theta_s}\right)^5\left(\frac{L_s}{100}\right)^5 -$$

$$\frac{5}{7}\left(\frac{D_2}{\theta_s}\right)^4\left(\frac{L_s}{100}\right)^4 + \frac{10}{8}\left(\frac{D_2}{\theta_s}\right)^3\left(\frac{L_s}{100}\right)^3 - \frac{10}{9}\left(\frac{D_2}{\theta_s}\right)^2\left(\frac{L_s}{100}\right)^2 +$$

$$\frac{5}{10}\ \frac{D_2}{\theta_s}\ \frac{L_s}{100} - \frac{1}{11}\Bigg]\Bigg], \qquad (14)$$

in which the angles are expressed in degrees. Next, proceeding as in Case I, we have

ϕ_2 = total deflection angle from A to B_1, that is ,

$$\phi_2 = \text{arc tan}\ \frac{y_1}{x_1}; \qquad (15)$$

$$\phi_1 = \Delta - \phi_2\ ; \qquad (16)$$

$$c_1 = \text{spiral "long chord"} = \frac{y_1}{\sin \phi_2}\ . \qquad (17)$$

$$T_1 = VB_1 = \text{long tangent} = \frac{y_1}{\sin \Delta}; \qquad (18)$$

$$T_2 = AV = \text{short tangent} = x_1 - y_1 \cot \Delta\ ; \qquad (19)$$

$$p = \frac{R_2 \text{ vers } \Delta_2 + 2R_1 \sin \frac{1}{2}\Delta_1 \sin\left(\frac{1}{2}\Delta_1 + \Delta_2\right) - y_1}{\cos \Delta_2} \qquad (20)$$

Eqs. (13-20) are basic formulas similar to Eqs. (1-8) of Case I.

<u>CONVENTIONAL METHOD</u>: Another set of formulas using nominal values of p (found in tables) will now be derived. Treating $AO_2O_1B_1A$ as a closed traverse we get expressions similar to Eqs. (9-12) in Case I, as follows:

$$N_2 = R_2 + (R_1 - R_2 - p) \cos \Delta_2 - R_1 \cos \Delta\ ; \qquad (p)$$

$$B_2 = R_1 \sin \Delta - (R_1 - R_2 - p) \sin \Delta_2\ ; \qquad (q)$$

$$\phi_2 = \text{total deflection angle from A to } B_1 = \text{arc tan}\ \frac{N_2}{B_2}\ ; \qquad (21)$$

$$\phi_1 = \Delta - \phi_2\ ; \qquad (22)$$

$$c = \frac{B_2}{\cos \phi_2}; \qquad (23)$$

$$T_1 = \frac{N_2}{\sin \Delta}; \qquad (24)$$

$$T_2 = B_2 - T_1 \cos\Delta = B_2 - N_2 \cot \Delta . \qquad (25)$$

This value of ϕ_2 should agree closely with the spiral deflection formula:

$$\frac{L_S D_2}{200} - \frac{\theta_S}{3}.$$

EXAMPLE 3. Given $D_1 = 6^0$, $D_2 = 10^0$, $L_S = 300$ ft. Required: The parts of a transition spiral (Fig. 2) by both the exact and the conventional method.

SOLUTION: $\theta_S = \frac{300}{200}(10-6) = 6^0$; $\theta_S/D_2 = \frac{1}{2}$.

EXACT METHOD: Applying Eqs. 13 and 14, we get x_1 = 290.17 ft., y_1 = 67.006 ft.; from which (Eqs. 15-20), c = 297.80 ft., $\phi_2 = 13^0 00'10''$, $\phi_1 = 10^0 59' 50''$, T_1 = 164.74 ft., T_2 = 139-67 ft., and p = 2.626 ft.

CONVENTIONAL METHOD: With $\theta_S = 6^0$ and L_S = 300 ft., the nominal value of p (from tables) = 2.616 ft. Accordingly the Eqs. 21-25, $\phi_2 = 13^0 00'09''$, $\phi_1 = 10^0 59'51''$, c = 297.86 ft., T_1 = 164.76 ft., and T_2 = 139.70 ft.

EXAMPLE 4. Given $D_1 = 4^0$, $D_2 = 6^0$, L_S = 600 ft. Required: The parts of a transition spiral (Fig. 2) by both the exact and the conventional method.

SOLUTION: $\theta_S = \frac{600}{200}(6-4) = 6^0$; $\theta_S/D_2 = 1$.

EXACT METHOD: Applying Eqs. 13 and 14, we get x_1 = 570.16 ft., y_1 = 163.541 ft.; from which $\phi_1 = 13^0 59'44''$, $\phi_2 = 16^0 00'16''$, c = 593.15 ft., T_1 = 327.08 ft., T_2 = 286.90 ft., and p = 4.995 ft.

CONVENTIONAL METHOD: With $\theta_S = 6^0$ and L_S = 600 ft., the nominal value of p (from tables) = 5.232 ft. Accordingly, by Eqs. 21-25, $\phi_2 = 16^0 00'13''$, $\phi_1 = 13^0 59'47''$, c = 593.26 ft., T_1 = 327.12 ft., and T_2 = 286.98 ft.

COMMENTS AND CONCLUSIONS

The four illustrative examples - two from each case - are intentionally more extreme than typical in order to obtain outside limits for purposes of comparison.

Examples 1 and 3 are identical except for the calculated values of x_1 and y_1, which are unequal because of different coordinate axes. In the first case, the x axis coincides with the initial tangent to the flatter curve, while in the second case, it coincides with the initial tangent to the sharper curve.

Formulas 1-8 (Case I) and 13-20 (Case II) are too cumbersome for use in the field but basic in compiling tables. Except for the nominal value of p, Formulas 9-12 (Case I) and 21-25 (Case II) are theoretically correct and in line with conventional methods giving results consistent with the accuracy of ordinary field measurements.

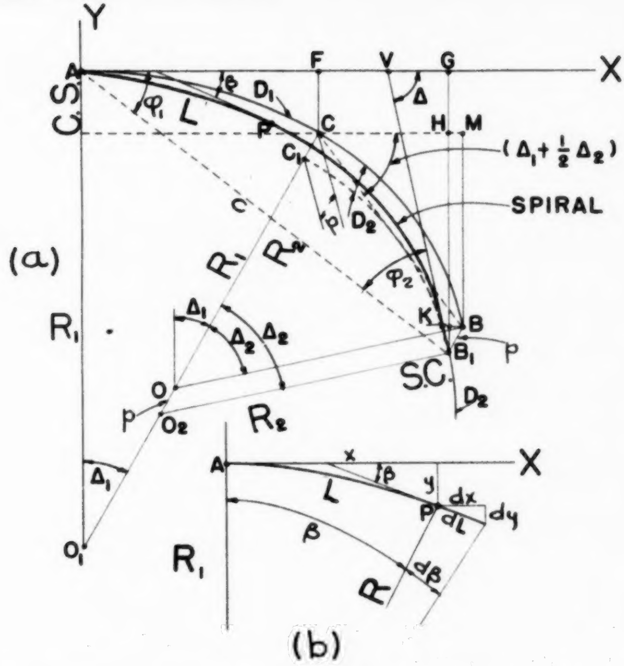

Fig. 1. Transition spiral between the branches of compound curve where the sharper branch is shifted radially inward at C (the P.C.C.).

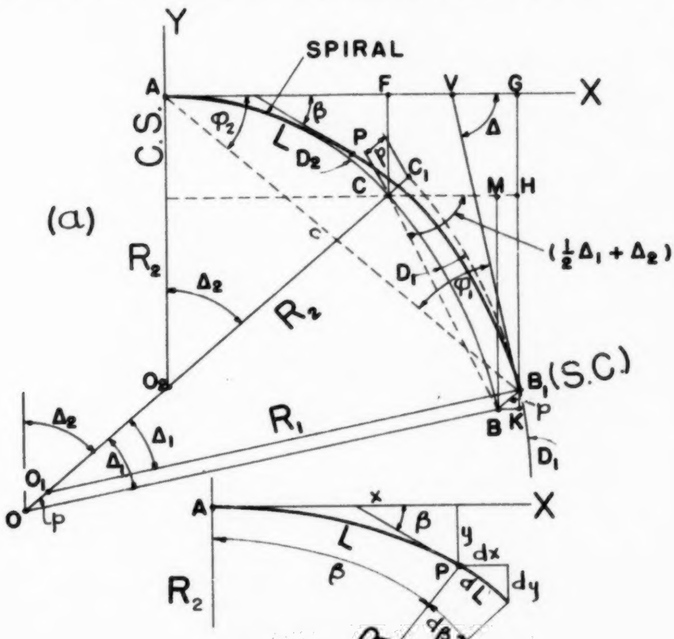

Fig. 2. Transition spiral between the branches of compound curve where the flatter branch is shifted radially outward at C (the P.C.C.).